INTRA REGULAR ABEL-GRASSMANN'S GROUPOIDS CHARACTERIZED BY THEIR INTUITIONISTIC FUZZY IDEALS

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Abstract. In this paper, we have discussed the properties of intuitionistic fuzzy ideals of an AG-groupoids. We have characterized an intra-regular AG-groupoid in terms of intuitionistic fuzzy left (right, two-sided) ideals, fuzzy (generalized) bi-ideals, intuitionistic fuzzy interior ideals and intuitionistic fuzzy quasi ideals. We have proved that the intuitionistic fuzzy left (right, interior, quasi) ideal coincides in an intra-regular AG-groupoid. We have also shown that the set of intuitionistic fuzzy two-sided ideals of an intra-regular AG-groupoid forms a semilattice structure.

Keywords. AG-groupoids, intra-regular AG-groupoids and intuitionistic fuzzy ideals.

Introduction

Given a set S, a fuzzy subset of S is an arbitrary mapping $f : S \to [0, 1]$ where $[0, 1]$ is the unit segment of a real line. This fundamental concept of fuzzy set was first given by Zadeh [16] in 1965. Fuzzy groups have been first considered by Rosenfeld [13] and fuzzy semigroups by Kuroki [9].

Atanassov [1], introduced the concept of an intuitionistic fuzzy set. Dengfeng and Chunfian [2] introduced the concept of the degree of similarity between intuitionistic fuzzy sets, which may be finite or continuous, and gave corresponding proofs of these similarity measure and discussed applications of the similarity measures between intuitionistic fuzzy sets to pattern recognition problems. Jun in [4], introduced the concept of an intuitionistic fuzzy bi-ideal in ordered semigroups and characterized the basic properties of ordered semigroups in terms of intuitionistic fuzzy bi-ideals. In [7] and [8], Kim and Jun introduced the concept of intuitionistic fuzzy interior ideals of semigroups. In [14], Shabir and Khan gave the concept of an intuitionistic fuzzy interior ideal of ordered semigroups and characterized different classes of ordered semigroups in terms of intuitionistic fuzzy interior ideals. They also gave the concept of an intuitionistic fuzzy generalized bi-ideal in [15] and discussed different classes of ordered semigroups in terms of intuitionistic fuzzy generalized bi-ideals.

In this paper, we consider the intuitionistic fuzzification of the concept of several ideals in AG-groupoid and investigate some properties of such ideals.

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An AG-groupoid is a non-associative algebraic structure mid way between a groupoid and a commutative semigroup. The left identity in an AG-groupoid if exists is unique [11]. An AG-groupoid is non-associative and non-commutative algebraic structure, nevertheless, it posses many interesting properties which we usually find in associative and commutative algebraic structures. An AG-groupoid with right identity becomes a commutative monoid [11]. An AG-groupoid is basically the generalization of semigroup (see [5]) with wide range of applications in theory of flocks [12]. The theory of flocks tries to describes the human behavior and interaction.

The concept of an Abel-Grassmann's groupoid (AG-groupoid) [5] was first given by M. A. Kazim and M. Naseeruddin in 1972 and they called it left almost semigroup (LA-semigroup). P. Holgate call it simple invertive groupoid [3]. An AG-groupoid is a groupoid having the left invertive law

$$(ab)c = (cb)a, \text{ for all } a,\ b,\ c \in S. \tag{1}$$

In an AG-groupoid, the medial law [5] holds

$$(ab)(cd) = (ac)(bd), \text{ for all } a,\ b,\ c,\ d \in S. \tag{2}$$

In an AG-groupoid S with left identity, the paramedial law holds

$$(ab)(cd) = (dc)(ba), \text{ for all } a, b, c, d \in S. \tag{3}$$

If an AG-groupoid contains a left identity, the following law holds

$$a(bc) = b(ac), \text{ for all } a,\ b,\ c \in S. \tag{4}$$

Preliminaries

Let S be an AG-groupoid, by an AG-subgroupoid of S, we means a non-empty subset A of S such that $A^2 \subseteq A$.

A non-empty subset A of an AG-groupoid S is called a left (right) ideal of S if $SA \subseteq A$ ($AS \subseteq A$).

A non-empty subset A of an AG-groupoid S is called a two-sided ideal or simply ideal if it is both a left and a right ideal of S.

A non empty subset A of an AG-groupoid S is called a generalized bi-ideal of S if $(AS)A \subseteq A$.

An AG-subgroupoid A of S is called a bi-ideal of S if $(AS)A \subseteq A$.

A non empty subset A of an AG-groupoid S is called an interior ideal of S if $(SA)S \subseteq A$.

A non empty subset A of an AG-groupoid S is called an quasi ideal of S if $AS \cap SA \subseteq A$.

A fuzzy subset f is a class of objects with a grades of membership having the form

$$f = \{(x,\ f(x))/x \in S\}.$$

An intuitionistic fuzzy set (briefly, IFS) A in a non empty set S is an object having the form

$$A = \{(x, \mu_A(x), \gamma_A(x))/x \in S\}.$$

The functions $\mu_A : S \longrightarrow [0,1]$ and $\gamma_A : S \longrightarrow [0,1]$ denote the degree of membership and the degree of nonmembership respectively such that for all $x \in S$, we have

$$0 \leq \mu_A(x) + \gamma_A(x) \leq 1.$$

For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for an IFS $A = \{(x, \mu_A(x), \gamma_A(x))/x \in S\}$.

Let $\delta = \{(x, S_\delta(x), \Theta_\delta(x))/S_\delta(x) = 1$ and $\Theta_\delta(x) = 0/x \in S\} = (S_\delta, \Theta_\delta)$ be an IFS, then $\delta = (S_\delta, \Theta_\delta)$ will be carried out in operations with an IFS $A = (\mu_A, \gamma_A)$ such that S_δ and Θ_δ will be used in collaboration with μ_A and γ_A respectively.

An IFS $A = (\mu_A, \gamma_A)$ of an AG-groupoid S is called an intuitionistic fuzzy AG-subgroupoid of S if $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\gamma_A(xy) \leq \gamma_A(x) \vee \gamma_A(y)$ for all x, $y \in S$.

An IFS $A = (\mu_A, \gamma_A)$ of an AG-groupoid S is called an intuitionistic fuzzy left ideal of S if $\mu_A(xy) \geq \mu_A(y)$ and $\gamma_A(xy) \leq \gamma_A(y)$ for all x, $y \in S$.

An IFS $A = (\mu_A, \gamma_A)$ of an AG-groupoid S is called an intuitionistic fuzzy right ideal of S if $\mu_A(xy) \geq \mu_A(x)$ and $\gamma_A(xy) \leq \gamma_A(x)$ for all x, $y \in S$.

An IFS $A = (\mu_A, \gamma_A)$ of an AG-groupoid S is called an intuitionistic fuzzy two-sided ideal of S if it is both an intuitionistic fuzzy left and an intuitionistic fuzzy right ideal of S.

An IFS $A = (\mu_A, \gamma_A)$ of an AG-groupoid S is called an intuitionistic fuzzy generalized bi-ideal of S if $\mu_A((xa)y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\gamma_A((xa)y) \leq \gamma_A(x) \vee \gamma_A(y)$ for all x, a and $y \in S$.

An intuitionistic fuzzy AG-subgroupoid $A = (\mu_A, \gamma_A)$ of an AG-groupoid S is called an intuitionistic fuzzy bi-ideal of S if $\mu_A((xa)y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\gamma_A((xa)y) \leq \gamma_A(x) \vee \gamma_A(y)$ for all x, a and $y \in S$.

An IFS $A = (\mu_A, \gamma_A)$ of an AG-groupoid S is called an intuitionistic fuzzy interior ideal of S if $\mu_A((xa)y) \geq \mu_A(a)$ and $\gamma_A((xa)y) \leq \gamma_A(a)$ for all x, a and $y \in S$.

An IFS $A = (\mu_A, \gamma_A)$ of an AG-groupoid S is called an intuitionistic fuzzy quasi ideal of S if $(\mu_A \circ S) \cap (S \circ \mu_A) \subseteq \mu_A$ and $(\gamma_A \circ S) \cup (S \circ \gamma_A) \supseteq \gamma_A$, that is, $(A \circ \delta) \cap (\delta \circ A) \subseteq A$.

Let S be an AG-groupoid and let $A_I = \{A/\ A \in S\}$, where $A = (\mu_A, \gamma_A)$ be any IFS of S, then $(A_I, \circ)$ satisfies (1), (2), (3) and (4).

An element a of an AG-groupoid S is called an intra-regular if there exist $x, y \in S$ such that $a = (xa^2)y$ and S is called an intra-regular if every element of S is an intra-regular.

Example 1. *Let $S = \{1,2,3,4,5\}$ be an AG-groupoid with left identity 4 with the following multiplication table.*

.	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	4	5	3
4	1	2	3	4	5
5	1	2	5	3	4

It is easy to see that S is an intra-regular. Define an IFS $A = (\mu_A, \gamma_A)$ of S as follows: $\mu_A(1) = 1$, $\mu_A(2) = \mu_A(3) = \mu_A(4) = \mu_A(5) = 0$, $\gamma_A(1) = 0.3$, $\gamma_A(2) = 0.4$

and $\gamma_A(3) = \gamma_A(4) = \gamma_A(5) = 0.2$, then clearly $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy two-sided ideal and also an intuitionistic fuzzy AG-subgroupoid of S.

For an IFS $A = (\mu_A, \gamma_A)$ of an AG-groupoid S and $\alpha \in (0, 1]$, the set

$$A_\alpha = \{x \in S : \mu_A(x) \geq \alpha,\ \gamma_A(x) \leq \alpha\}$$

is called an intuitionistic level cut of A.

Theorem 1. *For an AG-groupoid S, the following statements are true.*

(i) A_α is a right (left, two-sided) ideal of S if A is an intuitionistic fuzzy right (left) ideal of S but the converse is not true in general.

(ii) A_α is a bi-(generalized bi-) ideal of S if A is an intuitionistic fuzzy bi-(generalized bi-) ideal of S but the converse is not true in general.

Proof. (i): Let S be an AG-groupoid and let A be an intuitionistic fuzzy right ideal of S. If $x, y \in S$ such that $x \in A_\alpha$, then $\mu_A(x) \geq \alpha$ and $\gamma_A(x) \leq \alpha$ therefore $\mu_A(xy) \geq \mu_A(x) \geq \alpha$ and $\gamma_A(xy) \leq \gamma_A(x) \leq \alpha$. Thus $xy \in A_\alpha$, which shows that A_α is a right ideal of S. Let $y \in A_\alpha$, then $\mu_A(y) \geq \alpha$ and $\gamma_A(y) \leq \alpha$. If A is an intuitionistic fuzzy left ideal of S, then $\mu_A(xy) \geq \mu_A(y) \geq \alpha$ and $\gamma_A(xy) \leq \gamma_A(y) \leq \alpha$ implies that $xy \in A_\alpha$, which shows that A_α is a left ideal of S.

Conversely, let us define an IFS $A = (\mu_A, \gamma_A)$ of an AG-groupoid S in Example 1 as follows: $\mu_A(1) = 0.4$, $\mu_A(2) = 0.8$, $\mu_A(3) = \mu_A(4) = \mu_A(5) = 0$, $\gamma_A(1) = 0.4$, $\gamma_A(2) = 0.3$, $\gamma_A(3) = \gamma_A(4) = 0.9$ and $\gamma_A(5) = 1$. Let $\alpha = 0.4$, then it is easy to see that $A_\alpha = \{a, b\}$ and one can easily verify from Example 1 that $\{a, b\}$ is a right (left) ideal of S but $\mu_A(21) \ngeq \mu_A(2)$ $(\gamma_A(21) \nleq \gamma_A(2))$ and $\mu_A(12) \ngeq \mu_A(2)$ $(\gamma_A(12) \nleq \gamma_A(2))$ implies that A is not an intuitionistic fuzzy right (left) ideal of S.

(ii): Let S be an AG-groupoid and let A be an intuitionistic fuzzy bi-(generalized bi-) ideal of S. If x, y and $z \in S$ such that x and $z \in A_\alpha$, then $\mu_A(x) \geq \alpha$, $\gamma_A(x) \leq \alpha$, $\mu_A(z) \geq \alpha$ and $\gamma_A(z) \leq \alpha$. Therefore $\mu_A((xy)z) \geq \mu_A(x) \wedge \mu_A(z) \geq \alpha$ and $\gamma_A((xy)z) \leq \gamma_A(x) \vee \gamma_A(z) \leq \alpha$ implies that $(xy)z \in A_\alpha$. Which shows that A_α is a generalized bi-ideal of S. Now let $x, y \in A_\alpha$, then $\mu_A(x) \geq \alpha$, $\gamma_A(x) \leq \alpha$, $\mu_A(y) \geq \alpha$ and $\gamma_A(y) \leq \alpha$. Therefore $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y) \geq \alpha$ and $\gamma_A(xy) \leq \gamma_A(x) \vee \gamma_A(y) \leq \alpha$ implies that $xy \in A_\alpha$. Thus A_α is a bi-ideal of S.

Conversely, let us define an IFS $A = (\mu_A, \gamma_A)$ of an AG-groupoid S as in (i). Then it is easy to observe that $A_\alpha = \{a, b\}$ is a bi-(generalized bi-) ideal of S but $\mu_A((ba)b) \ngeq \mu_A(b)$ and $\gamma_A((ba)b) \nleq \gamma_A(b)$ implies that A is not an intuitionistic fuzzy bi-(generalized bi-) ideal of S. □

Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be any two $IFSs$ of an AG-groupoid S, then the product $A \circ B$ is defined by,

$$(\mu_A \circ \mu_B)(a) = \begin{cases} \bigvee\limits_{a=bc} \{\mu_A(b) \wedge \mu_B(c)\}, & \text{if } a = bc \text{ for some } b,\ c \in S. \\ 0, & \text{otherwise.} \end{cases}$$

$$(\gamma_A \circ \gamma_B)(a) = \begin{cases} \bigwedge\limits_{a=bc} \{\gamma_A(b) \vee \gamma_B(c)\}, & \text{if } a = bc \text{ for some } b,\ c \in S. \\ 1, & \text{otherwise.} \end{cases}$$

$A \subseteq B$ means that

$$\mu_A(x) \leq \mu_B(x) \text{ and } \gamma_A(x) \geq \gamma_B(x) \text{ for all } x \text{ in } S.$$

Lemma 1. ([10],[6]) *Let S be an AG-groupoid, then the following holds.*

(i) An IFS $A=(\mu_A,\gamma_A)$ is an intuitionistic fuzzy AG-subgroupoid of S if and only if $\mu_A\circ\mu_A\subseteq\mu_A$ and $\gamma_A\circ\gamma_A\supseteq\gamma_A$.
(ii) An IFS $A=(\mu_A,\gamma_A)$ is intuitionistic fuzzy left (right) ideal of S if and only if $S\circ\mu_A\subseteq\mu_A$ and $\Theta\circ\gamma_A\supseteq\gamma_A$ ($\mu_A\circ S\subseteq\mu_A$ and $\gamma_A\circ\Theta\supseteq\gamma_A$).

Theorem 2. *Let $A=(\mu_A,\gamma_A)$ be an IFS of an intra-regular AG-groupoid S with left identity, then the following conditions are equivalent.*

(i) $A=(\mu_A,\gamma_A)$ is an intuitionistic fuzzy bi-ideal of S.
(ii) $(A\circ\delta)\circ A=A$ and $A\circ A=A$, where $\delta=(S_\delta,\Theta_\delta)$.

Proof. $(i)\Longrightarrow(ii)$: Let $A=(\mu_A,\gamma_A)$ be an intuitionistic fuzzy bi-ideal of an intra-regular AG-groupoid S with left identity. Let $a\in A$, then there exists x, $y\in S$ such that $a=(xa^2)y$. Now by using (4), (1), (3) and (2), we have

$$\begin{aligned}
a &= (xa^2)y=(x(aa))y=(a(xa))y=(y(xa))a=(y(x((x(aa))y)))a\\
&= ((ey)(x((a(xa))y)))a=((((a(xa))y)x)(ye))a=(((xy)(a(xa)))(ye))a\\
&= ((a((xy)(xa)))(ye))a=((a(x^2(ya)))(ye))a=(((ye)(x^2(ya)))a)a\\
&= (((ye)(x^2(y((xa^2)(ey)))))a)a=(((ye)(x^2(y((ye)(a^2x)))))a)a\\
&= (((ye)(x^2(y(a^2((ye)x)))))a)a=(((ye)(x^2(a^2(y((ye)x)))))a)a\\
&= (((ye)(a^2(x^2(y((ye)x)))))a)a=(((aa)((ye)(x^2(y((ye)x)))))a)a\\
&= ((((x^2(y((ye)x)))(ye))(aa))a)a=((a(((x^2(y((ye)x)))(ye))a))a)a=(pa)a
\end{aligned}$$

where $p=a(((x^2(y((ye)x)))(ye))a)$. Therefore

$$\begin{aligned}
((\mu_A\circ S_\delta)\circ\mu_A)(a) &= \bigvee_{a=(pa)a}\{(\mu_A\circ S_\delta)(pa)\wedge\mu_A(a)\}\\
&\geq \bigvee_{pa=pa}\{\mu_A(p)\circ S_\delta(a)\}\wedge\mu_A(a)\\
&\geq \left\{\mu_A(a(((x^2(y((ye)x)))(ye))a))\wedge S_\delta(a)\right\}\wedge\mu_A(a)\\
&\geq \mu_A(a)\wedge 1\wedge\mu_A(a)=\mu_A(a)
\end{aligned}$$

and

$$\begin{aligned}
((\gamma_A\circ\Theta_\delta)\circ\gamma_A)(a) &= \bigwedge_{a=(pa)a}\{(\gamma_A\circ\Theta_\delta)(pa)\vee\gamma_A(a)\}\\
&\leq \bigwedge_{pa=pa}\{\gamma_A(p)\circ\Theta_\delta(a)\}\vee\gamma_A(a)\\
&\leq \left\{\gamma_A(a(((x^2(y((ye)x)))(ye))a))\vee\Theta_\delta(a)\right\}\vee\gamma_A(a)\\
&\leq \gamma_A(a)\vee 1\vee\gamma_A(a)=\gamma_A(a).
\end{aligned}$$

This shows that $(\mu_A\circ S_\delta)\circ\mu_A\supseteq\mu_A$ and $(\gamma_A\circ\Theta_\delta)\circ\gamma_A\subseteq\gamma_A$, which implies that $(A\circ\delta)\circ A\supseteq A$. Now by using (4), (1) and (3), we have

$$\begin{aligned}
a &= (xa^2)y=(x(aa))y=(a(xa))y=(y(xa))a=(y(x((xa^2)(ey))))a\\
&= (y(x((ye)(a^2x))))a=(y(x(a^2((ye)x))))a=(y(a^2(x((ye)x))))a\\
&= ((aa)(y(x((ye)x))))a=(((x((ye)x))y)(aa))a=(a((x((ye)x))a))a=(ap)a
\end{aligned}$$

where $p=((x((ye)x))y)a$. Therefore

$$\begin{aligned}((\mu_A \circ S_\delta)\circ \mu_A)(a) &= \bigvee_{a=(ap)a} \{(\mu_A \circ S_\delta)(ap) \wedge \mu_A(a)\} \\ &= \bigvee_{a=(ap)a} \left(\bigvee_{ap=ap} \mu_A(a) \wedge S_\delta(p)\right) \wedge \mu_A(a) \\ &= \bigvee_{a=(ap)a} \{\mu_A(a) \wedge 1 \wedge \mu_A(a)\} = \bigvee_{a=(ap)a} \mu_A(a) \wedge \mu_A(a) \\ &\leq \bigvee_{a=(ap)a} \mu_A((a((x((ye)x))a))a) = \mu_A(a).\end{aligned}$$

This shows that $(\mu_A \circ S_\delta)\circ \mu_A \subseteq \mu_A$ and similarly we can show that $(\gamma_A \circ \Theta_\delta)\circ \gamma_A \supseteq \gamma_A$, which implies that $(A\circ\delta)\circ A \subseteq A$. Thus $(A\circ\delta)\circ A = A$. We have shown that $a=((a(((x^2(y((ye)x)))(ye))a))a)a$. Let $a=pa$ where $p=(a(((x^2(y((ye)x)))(ye))a))a$. Therefore

$$\begin{aligned}(\mu_A \circ \mu_A)(a) &= \bigvee_{a=pa} \left\{\mu_A((a(((x^2(y((ye)x)))(ye))a))a) \wedge \mu_A(a)\right\} \\ &\geq \mu_A(a) \wedge \mu_A(a) \wedge \mu_A(a) = \mu_A(a).\end{aligned}$$

This shows that $\mu_A \circ \mu_A \supseteq \mu_A$ and similarly we can show that $\gamma_A \circ \gamma_A \subseteq \gamma_A$. Now by using Lemma 1, we get $A \circ A = A$.

$(ii) \Longrightarrow (i)$: Let $A=(\mu_A,\gamma_A)$ be an IFS of an intra-regular AG-groupoid S, then

$$\begin{aligned}\mu_A((xa)y) &= ((\mu_A \circ S_\delta)\circ \mu_A)((xa)y) = \bigvee_{(xa)y=(xa)y} \{(\mu_A \circ S_\delta)(xa) \wedge \mu_A(y)\} \\ &\geq \bigvee_{xa=xa} \{\mu_A(x) \wedge S_\delta(a)\} \wedge \mu_A(y) \geq \mu_A(x) \wedge 1 \wedge \mu_A(y) = \mu_A(x) \wedge \mu_A(z).\end{aligned}$$

This shows that $\mu_A((xa)y) \geq \mu_A(x) \wedge \mu_A(z)$ and similarly we can show that $\gamma_A((xa)y) \leq \gamma_A(x) \vee \gamma_A(z)$. Also by Lemma 1, A is an intuitionistic fuzzy AG-subgroupoid of S and therefore A is an intuitionistic fuzzy bi-ideal of S. $\square$

Theorem 3. *Let $A=(\mu_A,\gamma_A)$ be an IFS of an intra-regular AG-groupoid S with left identity, then the following conditions are equivalent.*

(i) $A=(\mu_A,\gamma_A)$ is an intuitionistic fuzzy interior ideal of S.
(ii) $(\delta\circ A)\circ\delta = A$, where $\delta=(S_\delta,\Theta_\delta)$.

Proof. $(i) \Longrightarrow (ii)$: Let $A=(\mu_A,\gamma_A)$ be an intuitionistic fuzzy interior ideal of an intra-regular AG-groupoid S with left identity. Let $a\in A$, then there exists x, $y\in S$ such that $a=(xa^2)y$. Now by using (4), (3)and (1), we have

$$a=(x(aa))y=(a(xa))y=((ea)(xa))y=((ax)(ae))y=(((ae)x)a)y.$$

Therefore

$$\begin{aligned}((S_\delta \circ \mu_A)\circ S_\delta)(a) &= \bigvee_{a=(((ae)x)a)y} \{(S_\delta \circ \mu_A)(((ae)x)a) \wedge S_\delta(y)\} \\ &\geq \bigvee_{((ae)x)a=((ae)x)a} \{(S_\delta((ae)x) \wedge \mu_A(a)\} \wedge 1 \\ &\geq 1 \wedge \mu_A(a) \wedge 1 = \mu_A(a).\end{aligned}$$

This proves that $(S_\delta \circ \mu_A) \circ S_\delta \supseteq \mu_A$ and similarly we can show that $(\Theta_\delta \circ \gamma_A) \circ \Theta_\delta \subseteq \gamma_A$, therefore $(\delta \circ A) \circ \delta \supseteq A$. Now again

$$\begin{aligned} ((S_\delta \circ \mu_A) \circ S_\delta)(a) &= \bigvee_{a=(xa^2)y} \{(S_\delta \circ \mu_A)(xa^2) \wedge S_\delta(y)\} \\ &= \bigvee_{a=(xa^2)y} \left(\bigvee_{xa^2=xa^2} S_\delta(x) \wedge \mu_A(a^2) \right) \wedge S_\delta(y) \\ &= \bigvee_{a=(xa^2)y} \{1 \wedge \mu_A(a^2) \wedge 1\} = \bigvee_{a=(xa^2)y} \mu_A(a^2) \\ &\leq \bigvee_{a=(xa^2)y} \mu_A((xa^2)y) = \mu_A(a). \end{aligned}$$

Thus $(S_\delta \circ \mu_A) \circ S_\delta \subseteq \mu_A$ and similarly we can show that $(\Theta_\delta \circ \gamma_A) \circ \Theta_\delta \supseteq \gamma_A$, therefore $(\delta \circ A) \circ \delta \subseteq A$. Hence it follows that $(\delta \circ A) \circ \delta = A$.

$(ii) \Longrightarrow (i)$: Let $A = (\mu_A, \gamma_A)$ be an IFS of an intra-regular AG-groupoid S, then

$$\begin{aligned} \mu_A((xa)y) &= ((S_\delta \circ \mu_A) \circ S_\delta)((xa)y) = \bigvee_{(xa)y=(xa)y} \{(S_\delta \circ \mu_A)(xa) \wedge S_\delta(y)\} \\ &\geq \bigvee_{xa=xa} \{(S_\delta(x) \circ \mu_A(a)\} \wedge S_\delta(y) \geq \mu_A(a). \end{aligned}$$

Similarly we can show that $\gamma_A((xa)y) \leq \gamma_A(a)$ and therefore $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy interior ideal of S. $\square$

Lemma 2. *Let $A = (\mu_A, \gamma_A)$ be an IFS of an intra-regular AG-groupoid S with left identity, then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy left ideal of S if and only if $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right ideal of S.*

Proof. Let S be an intra-regular AG-groupoid and let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy left ideal of S. Now for $a, b \in S$ there exists $x, y, x', y' \in S$ such that $a = (xa^2)y$ and $b = (x'b^2)y'$, then by using (1), (3) and (4), we have

$$\begin{aligned} \mu_A(ab) &= \mu_A(((xa^2)y)b) = \mu_A((by)(x(aa))) = \mu_A(((aa)x)(yb)) \\ &= \mu_A(((xa)a)(yb)) = \mu_A(((xa)(ea))(yb)) = \mu_A(((ae)(ax))(yb)) \\ &= \mu_A((a((ae)x))(yb)) = \mu_A(((yb)((ae)x))a) \geq \mu_A(a). \end{aligned}$$

Similarly we can get $\gamma_A(ab) \leq \gamma_A(a)$, which implies that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right ideal of S.

Conversely let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy right ideal of S. Now by using (4) and (3), we have

$$\begin{aligned} \mu_A(ab) &= \mu_A(a((x'b^2)y') = \mu_A((x'b^2)(ay')) = \mu_A((y'a)(b^2x')) \\ &= \mu_A(b^2((y'a)x)) \geq \mu_A(b). \end{aligned}$$

Also we can get $\gamma_A(ab) \leq \gamma_A(b)$, which implies that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy left ideal of S. $\square$

An AG-groupoid S is called a left (right) duo if every left (right) ideal of S is a two-sided ideal of S and is called a duo if it is both a left and a right duo.

An AG-groupoid S is called an intuitionistic fuzzy left (right) duo if every intuitionistic fuzzy left (right) ideal of S is an intuitionistic fuzzy two-sided ideal of S

and is called an intuitionistic fuzzy duo if it is both an intuitionistic fuzzy left and an intuitionistic fuzzy right duo.

Corollary 1. *Every intra-regular AG-groupoid with left identity is an intuitionistic fuzzy duo.*

Let S be an AG-groupoid and let $\emptyset \neq A \subseteq S$ be an IFS of S, then the intuitionistic characteristic function $\chi_A = (\mu_{\chi_A}, \gamma_{\chi_A})$ of A is defined as

$$\mu_{\chi_A}(x) = \begin{cases} 1, \text{ if } x \in A \\ 0, \text{ if } x \notin A \end{cases} \text{ and } \gamma_{\chi_A}(x) = \begin{cases} 0, \text{ if } x \in A \\ 1, \text{ if } x \notin A \end{cases}$$

It is clear that γ_{χ_A} acts as a complement of μ_{χ_A}, that is, $\gamma_{\chi_A} = \mu_{\chi_{A^C}}$.

Lemma 3. ([10],[6]) *For any subset A of an AG-groupoid S, the following properties holds.*

(i) A is an AG-subgroupoid of S if and only if χ_A is an intuitionistic fuzzy AG-subgroupoid of S.

(ii) A is a fuzzy left (right, two-sided) ideal of S if and only if χ_A is an intuitionistic fuzzy left (right, two-sided) ideal of S.

Theorem 4. *An intra-regular AG-groupoid S with left identity is a left (right) duo if and only if it is an intuitionistic fuzzy left (right) duo.*

Proof. Let an intra-regular AG-groupoid S be a left duo and let $A = (\mu_A, \gamma_A)$ be any intuitionistic fuzzy left ideal of S. Let $a, b \in S$, then $a \in (Sa^2)S$. Now as Sa is a left ideal of S, therefore by hypothesis, Sa is a two-sided ideal of S. Now by using (4) and (1), we have

$$ab \in ((Sa^2)S)b = ((S(aa))S)b = ((a(Sa))S)b = ((S(Sa))a)b \subseteq ((S(Sa))S)S \subseteq Sa.$$

Thus $ab = ca$ for some $c \in S$. Now $\mu_A(ab) = \mu_A(ca) \geq \mu_A(a)$ and similarly $\gamma_A(ab) = \gamma_A(ca) \leq \gamma_A(a)$ implies that A is an intuitionistic fuzzy right ideal of S and therefore S is an intuitionistic fuzzy left duo.

Conversely, assume that S is a fuzzy left duo and L is any left ideal of S. Now by Lemma 3, the intuitionistic characteristic function $\chi_L = (\mu_{\chi_L}, \gamma_{\chi_L})$ of L is an intuitionistic fuzzy left ideal of S. Thus by hypothesis χ_L is an intuitionistic fuzzy two-sided ideal of S and by using Lemma 3, L is a two-sided ideal of S. Thus S is a left duo.

Now again let S be an intra-regular AG-groupoid such that S is a right duo and assume that $A = (\mu_A, \gamma_A)$ is any fuzzy right ideal of S. Clearly b^2S is a right ideal and so is a two-sided ideal of S. Let $a, b \in S$, then there exist $x, y \in S$ such that $b = (xb^2)y$. Now by using (3), we have

$$ab = a((xb^2)y) = a(((ex)(eb^2))y) = a(((b^2e)(xe))y) \subseteq S(((b^2S)S)S) \subseteq b^2S.$$

Thus $ab = (bb)c$ for some $c \in S$. Now $\mu_A(ab) = \mu_A((bb)c) \geq \mu_A(b)$ and $\gamma_A(ab) = \gamma_A((bb)c) \leq \gamma_A(b)$ implies that A is an intuitionistic fuzzy left ideal of S and therefore S is an intuitionistic fuzzy right duo. The Converse is simple. $\square$

Lemma 4. *In an intra-regular AG-groupoid S, $\delta \circ A = A$ and $A \circ \delta = A$ holds for an IFS $A = (\mu_A, \gamma_A)$ of S where $\delta = (S_\delta, \Theta_\delta)$.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an IFS of an intra-regular AG-groupoid S and let $a \in S$, then there exist $x \in S$ such that $a = (xa^2)y$. Now by using (4) and (1), we have

$$a = (x(aa))y = (a(xa))y = (y(xa))a.$$

Therefore

$$\begin{aligned}(S_\delta \circ \mu_A)(a) &= \bigvee_{a=(y(xa))a} \{S_\delta(y(xa)) \wedge \mu_A(a)\} = \bigvee_{a=(y(xa))a} \{1 \wedge \mu_A(a)\} \\ &= \bigvee_{a=(y(xa))a} \mu_A(a) = \mu_A(a).\end{aligned}$$

Similarly we can show that $\Theta_\delta \circ \gamma_A = \gamma_A$, which shows that $\delta \circ A = A$.
Now by using (3) and (4), we have

$$a = (xa^2)(ey) = (ye)(a^2x) = (aa)((ye)x) = (x(ye))(aa) = a((x(ye))a).$$

Therefore

$$\begin{aligned}(\mu_A \circ S_\delta)(a) &= \bigvee_{a=a((x(ye))a)} \{\mu_A(a) \wedge S_\delta((x(ye))a)\} = \bigvee_{a=a((x(ye))a)} \{\mu_A(a) \wedge 1\} \\ &= \bigvee_{a=a((x(ye))a)} \mu_A(a) = \mu_A(a).\end{aligned}$$

Similarly we can show that $\gamma_A \circ \Theta_\delta = \gamma_A$ which shows that $A \circ \delta = A$. □

Corollary 2. *In an intra-regular AG-groupoid S, $\delta \circ A = A$ and $A \circ \delta = A$ holds for every intuitionistic fuzzy left (right, two-sided) $A = (\mu_A, \gamma_A)$ of S, where $\delta = (S_\delta, \Theta_\delta)$.*

Lemma 5. *In an intra-regular AG-groupoid S, $\delta \circ \delta = \delta$, where $\delta = (S_\delta, \Theta_\delta)$.*

Proof. Let S be an intra-regular AG-groupoid, then

$$(S_\delta \circ S_\delta)(a) = \bigvee_{a=(xa^2)y} \{S_\delta(xa^2) \wedge S_\delta(y)\} = 1 = S_\delta(a)$$

and

$$(\Theta_\delta \circ \Theta_\delta)(a) = \bigwedge_{a=(xa^2)y} \{\Theta_\delta(xa^2) \vee \Theta_\delta(y)\} = 0 = \Theta_\delta(a).$$

□

Theorem 5. *Let $A = (\mu_A, \gamma_A)$ be an IFS of an intra-regular AG-groupoid S with left identity, then the following conditions are equivalent.*

(i) $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy quasi ideal of S.
(ii) $(A \circ \delta) \cap (\delta \circ A) = A$, where $\delta = (S_\delta, \Theta_\delta)$.

Proof. $(i) \Longrightarrow (ii)$ can be followed from Lemma 4 and $(ii) \Longrightarrow (i)$ is obvious. □

Theorem 6. *Let $A = (\mu_A, \gamma_A)$ an IFS of an intra-regular AG-groupoid S with left identity, then the following statements are equivalent.*

(i) A is an intuitionistic fuzzy two-sided ideal of S.
(ii) A is an intuitionistic fuzzy quasi ideal of S.

Proof. $(i) \Longrightarrow (ii)$ is an easy consequence of Corollary 2 and Theorem 5.

$(ii) \Longrightarrow (i)$: Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy quasi ideal of an intra-regular AG-groupoid S with left identity and let $a \in S$, then there exist $x \in S$ such that $a = (xa^2)y$. Now by using (4) and (3), we have

$$\begin{aligned} a &= (x(aa))y = (a(xa))(ey) = (ye)((xa)(ea)) = (ye)((ae)(ax)) \\ &= (ye)(a((ae)x)) = a((ye)((ae)x)). \end{aligned}$$

Therefore

$$\begin{aligned} (\mu_A \circ S_\delta)(a) &= \bigvee_{a=a((ye)((ae)x))} \{\mu_A(a) \wedge S_\delta((ye)((ae)x))\} \\ &\geq \mu_A(a) \wedge 1 = \mu_A(a). \end{aligned}$$

Similarly we can show that $\gamma_A \circ \Theta_\delta \subseteq \gamma_A$ which implies that $A \circ \delta \supseteq A$. Now by using Lemmas 4, 5 and (2), we have

$$A \circ \delta = (\delta \circ A) \circ (\delta \circ \delta) = (\delta \circ \delta) \circ (A \circ \delta) = \delta \circ (A \circ \delta) \supseteq \delta \circ A.$$

This shows that $\delta \circ A \subseteq (A \circ \delta) \cap (\delta \circ A)$. As A is an intuitionistic fuzzy quasi ideal of S, thus we get $\delta \circ A \subseteq A$. Now by using Lemma 1, A is an intuitionistic fuzzy left ideal of S and by Lemma 2, A is an intuitionistic fuzzy right ideal of S, that is, A is an intuitionistic fuzzy two-sided ideal of S. □

Theorem 7. *Let $A = (\mu_A, \gamma_A)$ be an IFS of an intra-regular AG-groupoid S with left identity, then the following statements are equivalent.*

(i) A is an intuitionistic fuzzy two-sided ideal of S.
(ii) A is an intuitionistic fuzzy interior ideal of S.

Proof. $(i) \Longrightarrow (ii)$ is obvious.

$(ii) \Longrightarrow (i)$: Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy interior ideal of an intra-regular AG-groupoid S with left identity and let $a, b \in S$, then there exist $x \in S$ such that $a = (xa^2)y$. Now by using (4), (1) and (3), we have

$$\begin{aligned} \mu_A(ab) &= \mu_A(((x(aa))y)b) = \mu_A(((a(xa))y)b) = \mu_A((by)(a(xa))) \\ &= \mu_A(((xa)a)(yb)) \geq \mu_A(a). \end{aligned}$$

Similarly we can prove that $\gamma_A(ab) \leq \gamma_A(a)$. Thus A is an intuitionistic fuzzy right ideal of S and by using Lemma 2, A is an intuitionistic fuzzy two-sided ideal of S. □

Theorem 8. *Let $A = (\mu_A, \gamma_A)$ be an IFS of an intra-regular AG-groupoid S with left identity, then the following statements are equivalent.*

(i) A is an intuitionistic fuzzy left ideal of S.
(ii) A is an intuitionistic fuzzy right ideal of S.
(iii) A is an intuitionistic fuzzy two-sided ideal of S.
(iv) A is an intuitionistic fuzzy bi-ideal of S.
(v) A is an intuitionistic fuzzy generalized bi-ideal of S.
(vi) A is an intuitionistic fuzzy interior ideal of S.
(vii) A is an intuitionistic fuzzy quasi ideal of S.
$(viii)$ $A \circ \delta = A$ and $\delta \circ A = A$.

Proof. $(i) \Longrightarrow (viii)$ can be followed from Corollary 2 and $(ix) \Longrightarrow (viii)$ is obvious.

$(vii) \Longrightarrow (vi)$: Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy quasi ideal of an intra-regular AG-groupoid S with left identity. Now for $a \in S$ there exist $x, y \in S$ such that $a = (ba^2)c$. Now by using (4), (3) and (1), we have

$$\begin{aligned} (xa)y &= (x((ba^2)c))y = ((ba^2)(xc))y = ((cx)(a^2b))y = (a^2((cx)b))y \\ &= (y((cx)b))(aa) = a((y((cx)b))a) \end{aligned}$$

and

$$\begin{aligned} (xa)y &= (x((ba^2)c))y = ((ba^2)(xc))y = ((cx)(a^2b))y = (a^2((cx)b))y \\ &= (y((cx)b))(aa) = (aa)(((cx)b)y) = ((((cx)b)y)a)a. \end{aligned}$$

Now by using Theorem 5, we have

$$\mu_A((xa)y) = ((\mu_A \circ S_\delta) \cap (S_\delta \circ \mu_A))((xa)y) = (\mu_A \circ S_\delta)((xa)y) \wedge (S_\delta \circ \mu_A)((xa)y).$$

Now

$$(\mu_A \circ S_\delta)((xa)y) = \bigvee_{(xa)y = a((y((cx)b))a)} \{\mu_A(a) \wedge S_\delta((y((cx)b))a)\} \geq \mu_A(a)$$

and

$$(S_\delta \circ \mu_A)((xa)y) = \bigvee_{(xa)y = ((((cx)b)y)a)a} \{S_\delta((((cx)b)y)a) \wedge \mu_A(a)\} \geq \mu_A(a).$$

This implies that $\mu_A((xa)y) \geq \mu_A(a)$ and similarly we can show that $\gamma_A((xa)y) \leq \gamma_A(a)$. Thus A is an intuitionistic fuzzy interior ideal of S.

$(vi) \Longrightarrow (v)$: Let A be an intuitionistic fuzzy interior ideal of S, then by Theorem 7, A is an intuitionistic fuzzy two-sided ideal of S and it is easy to observe that A is an intuitionistic fuzzy generalized bi-ideal of S.

$(v) \Longrightarrow (iv)$: Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy generalized bi-ideal of an intra-regular AG-groupoid S with left identity . Let $a \in S$, then there exists $x, y \in S$ such that $a = (xa^2)y$. Now by using (4), (3) and (1), we have

$$\begin{aligned} \mu_A(ab) &= \mu_A(((x(aa))y)b) = \mu_A((((ea)(xa))y)b) = \mu_A((((ax)(ae))y)b) \\ &= \mu_A(((a((ax)e))(ey))b) = \mu_A(((ye)(((ax)e)a))b) \\ &= \mu_A(((ye)((ae)(ax)))b) = \mu_A(((ye)(a((ae)x)))b) \\ &= \mu_A((a((ye)((ae)x)))b) \geq \mu_A(a) \wedge \mu_A(b). \end{aligned}$$

Similarly we can show that $\gamma_A(ab) \leq \gamma_A(a) \vee \gamma_A(b)$ and therefore $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy bi-ideal of S.

$(iv) \Longrightarrow (iii)$: Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy bi-ideal of an intra-regular AG-groupoid S with left identity. Let $a \in S$, then there exists $x, y \in S$ such that $a = (xa^2)y$. Now by using (4), (1) and (3), we have

$$\begin{aligned} \mu_A(ab) &= \mu_A(((x(aa))y)b) = \mu_A(((a(xa))y)b) = \mu_A((by)((ea)(xa))) \\ &= \mu_A((by)((ax)(ae))) = \mu_A(((ae)(ax))(yb)) = \mu_A((a((ae)x))(yb)) \\ &= \mu_A(((yb)((ae)x))a) = \mu_A(((yb)((((xa^2)y)e)x))a) \\ &= \mu_A(((yb)((y(xa^2))(ex)))a) = \mu_A(((yb)((xe)((xa^2)(ey))))a) \\ &= \mu_A(((yb)((xe)((ye)(a^2x))))a) = \mu_A(((yb)((xe)(a^2((ye)x))))a) \\ &= \mu_A(((yb)(a^2((xe)((ye)x))))a) = \mu_A((a^2((yb)((xe)((ye)x))))a) \\ &\geq \mu_A(a^2) \wedge \mu_A(a) \geq \mu_A(a) \wedge \mu_A(a) \wedge \mu_A(a) = \mu_A(a). \end{aligned}$$

Similarly we can prove that $\gamma_A(ab) \leq \gamma_A(a)$ and therefore $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right ideal of S. Now by using Lemma 2, $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy two-sided ideal of S.

$(iii) \Longrightarrow (ii)$ and $(ii) \Longrightarrow (i)$ are an easy consequences of Lemma 2. $\square$

Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ are $IFSs$ of an AG-groupoid S. The symbols $A \cap B$ will means the following IFS of S

$(\mu_A \cap \mu_B)(x) = \min\{\mu_A(x), \mu_B(x)\} = \mu_A(x) \wedge \mu_B(x)$, for all x in S.
$(\gamma_A \cup \gamma_B)(x) = \max\{\gamma_A(x), \gamma_B(x)\} = \gamma_A(x) \vee \gamma_B(x)$, for all x in S.

The symbols $A \cup B$ will means the following IFS of S

$(\mu_A \cup \mu_B)(x) = \max\{\mu_A(x), \mu_B(x)\} = \mu_A(x) \vee \mu_B(x)$, for all x in S.
$(\gamma_A \cap \gamma_B)(x) = \min\{\gamma_A(x), \gamma_B(x)\} = \gamma_A(x) \wedge \gamma_B(x)$, for all x in S.

Lemma 6. *Let S be an intra-regular AG-groupoid with left identity and let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ are any intuitionistic fuzzy two-sided ideals of S, then $A \circ B = A \cap B$.*

Proof. Assume that $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ are any intuitionistic fuzzy two-sided ideals of an intra-regular AG-groupoid S with left identity, then by using Lemma 1, we have $\mu_A \circ \mu_B \subseteq \mu_A \cap \mu_B$ and $\gamma_A \circ \gamma_B \supseteq \gamma_A \cup \gamma_B$, which shows that $A \circ B \subseteq A \cap B$. Let $a \in S$, then there exists $x, y \in S$ such that $a = (xa^2)y$. Now by using (4) and (2), we have

$$a = (x(aa))y = (a(xa))(ey) = (ae)((xa)y).$$

Therefore, we have

$$\begin{aligned}(\mu_A \circ \mu_B)(a) &= \bigvee_{a=(ae)((xa)y)} \{\mu_A(ae) \wedge \mu_B((xa)y)\} \geq \mu_A(ae) \wedge \mu_B((xa)y) \\ &\geq \mu_A(a) \wedge \mu_B(a) = (\mu_A \cap \mu_B)(a)\end{aligned}$$

and

$$\begin{aligned}(\gamma_A \circ \gamma_A)(a) &= \bigwedge_{a=(ae)((xa)y)} \{\gamma_A(ae) \vee \gamma_A((xa)y)\} \leq \gamma_A(ae) \vee \gamma_A((xa)y) \\ &\leq \gamma_A(a) \vee \gamma_A(a) = (\gamma_A \cup \gamma_A)(a).\end{aligned}$$

Thus we get that $\mu_A \circ \mu_B \supseteq \mu_A \cap \mu_B$ and $\gamma_A \circ \gamma_B \subseteq \gamma_A \cup \gamma_B$, which give us $A \circ B \supseteq A \cap B$ and therefore $A \circ B = A \cap B$. $\square$

The converse of Lemma 6 is not true in general which is discussed in the following.

Let us consider an AG-groupoid $S = \{1, 2, 3, 4, 5\}$ with left identity 4 in the following Cayley's table.

.	1	2	3	4	5
1	1	1	1	1	1
2	1	5	5	3	5
3	1	5	5	2	5
4	1	2	3	4	5
5	1	5	5	5	5

Define an IFS $A = (\mu_A, \gamma_A)$ of an AG-groupoid S as follows: $\mu_A(1) = \mu_A(2) = \mu_A(3) = 0.3$, $\mu_A(4) = 0.1$, $\mu_A(5) = 0.4$, $\gamma_A(1) = 0.2$, $\gamma_A(2) = 0.3$, $\gamma_A(3) = 0.4$, $\gamma_A(4) = 0.5$, $\gamma_A(5) = 0.2$. Now again define an IFS $B = (\mu_B, \gamma_B)$ of an AG-groupoid S as follows: $\mu_B(1) = \mu_B(2) = \mu_B(3) = 0.5$, $\mu_B(4) = 0.4$, $\mu_B(5) = 0.6$,

$\gamma_B(1) = 0.3$, $\gamma_B(2) = 0.4$, $\gamma_B(3) = 0.5$, $\gamma_B(4) = 0.6$, $\gamma_B(5) = 0.3$. Then it is easy to observe that $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ are an intuitionistic fuzzy two-sided ideals of S such that $(\mu_A \circ \mu_B)(a) = \{0.1,\ 0.3,\ 0.4\} = (\mu_A \cap \mu_B)(a)$ for all $a \in S$ and similarly $(\gamma_A \circ \gamma_B)(a) = (\gamma_A \cap \gamma_B)$ for all $a \in S$, that is, $A \circ B = A \cap B$ but S is not an intra-regular because $3 \in S$ is not an intra-regular.

An IFS $A = (\mu_A, \gamma_A)$ of an AG-groupoid is said to be idempotent if $\mu_A \circ \mu_A = \mu_A$ and $\gamma_A \circ \gamma_A = \gamma_A$, that is, $A \circ A = A$ or $A^2 = A$.

Lemma 7. *Every intuitionistic fuzzy two-sided ideal $A = (\mu_A, \gamma_A)$ of an intra-regular AG-groupoid S is idempotent.*

Proof. Let S be an intra-regular AG-groupoid and let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy two-sided ideal of S. Now for $a \in S$ there exists $x, y \in S$ such that $a = (xa^2)y$. Now by using (4) and (2), we have

$$a = (x(aa))y = (a(xa))(ey) = (ae)((xa)y).$$

$$\begin{aligned}(\mu_A \circ \mu_A)(a) &= \bigvee_{a=(ae)((xa)y)} \{\mu_A(ae) \wedge \mu_A((xa)y)\} \geq \mu_A(ae) \wedge \mu_A((xa)y) \\ &\geq \mu_A(a) \wedge \mu_A(a) = \mu_A(a).\end{aligned}$$

This shows that $\mu_A \circ \mu_A \supseteq \mu_A$ and by using Lemma 1, $\mu_A \circ \mu_A \subseteq \mu_A$, therefore $\mu_A \circ \mu_A = \mu_A$. Similarly we can prove that $\gamma_A \circ \gamma_A = \gamma_A$, which implies that $A = (\mu_A, \gamma_A)$ is idempotent. $\square$

Theorem 9. *The set of intuitionistic fuzzy two-sided ideals of an intra-regular AG-groupoid S forms a semilattice structure with identity δ, where $\delta = (S_\delta, \Theta_\delta)$.*

Proof. Let $\mathbb{I}_{\mu\gamma}$ be the set of intuitionistic fuzzy two-sided ideals of an intra-regular AG-groupoid S and let $A = (\mu_A, \gamma_A)$, $B = (\mu_B, \gamma_B)$ and $C = (\mu_C, \gamma_C)$ are any intuitionistic fuzzy two-sided ideals of $\mathbb{I}_{\mu\gamma}$. Clearly $\mathbb{I}_{\mu\gamma}$ is closed and by Lemma 7, we have $A^2 = A$. Now by using Lemma 6, we get $A \circ B = B \circ A$ and therefore, we have

$$(A \circ B) \circ C = (B \circ A) \circ C = (C \circ A) \circ B = (A \circ C) \circ B = (B \circ C) \circ A = A \circ (B \circ C).$$

It is easy to see from Corollary 2 that δ is an identity in $\mathbb{I}_{\mu\gamma}$. $\square$

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